Int. J. Heat Mass Transfer. Vol. 16, pp. 1097–1101. Pergamon Press 1973. Printed in Great Britain

SOME RELATIONS FOR THE ULTRASONIC REGION OF FLOW OF A REAL GAS IN THE PRESENCE OF HEAT TRANSFER

E. A. ORUDZHALIEV
Azerbaijan Institute of Petroleum and Chemistry, Baku, USSR

(*Received* 20 *July* 1970)

Abstract—Design relations for the flow of a real gas in the presence of heat transfer are derived on the basis of A. A. Gukhman's entropy method.

NOMENCLATURE

p, flow pressure [N/m^2];
T, flow temperature [°K];
T_0, stagnation temperature [°K];
T_{cr}, critical flow temperature [°K];
U, specific gas volume [m^3/kg];
R, gas constant [J/kg.K];
z, compressibility coefficient;
c_p, mass heat capacity of a real gas at constant pressure, [J/kg.K];
c_v, mass heat capacity of a real gas at constant volume [J/kg.K];
x, adiabatic index of a real gas;
k, adiabatic index of an ideal gas;
a, sound velocity in a real gas [m/s];
y, ratio of sound velocities in a real gas and in an ideal gas (at constant temperature);
w, flow velocity [m/s];
G, gas flow rate per second [m/s];
a_{cr}, critical velocity of a real gas [m/s];
ξ_{cr}, ratio of critical velocities of real and ideal gases (at a constant temperature T_0);
λ, w/a_{cr};
ρ, gas density [kg/m^3];
σ, reduced entropy;
F_{cr}, critical section [m^2];
τ, T/T_{cr};
f, F/F_{cr}.

THE AIM of the present paper is to find out a number of relations for the ultrasonic region of flow of a real gas in the presence of heat transfer. To solve the problems stated, A. A. Gukhman's very fruitful entropy method based on the hypothesis of a linear entropy approximation is employed. These problems related to the solution of a number of hydrodynamic problems on the flow of an ideal gas are considered in detail in [1, 2].

In the present theoretical study the derivation and relations themselves naturally become complex due to the fact that in the thermodynamic sense the moving gas is considered to be real, and the heat transfer factor is still taken into account.

Here, in a non-adiabatic flow in addition to the effect of friction and geometry there also appears the effect of heat transfer.

Within the framework of a thermodynamic analysis it is possible to say only that thermal effects mostly manifest themselves weakly since the amount of heat enters into the perturbing function with a multiplier (k-1).

Convective heat transfer in our case may be studied by using the Reynolds analogy.

In [3] it is shown that under the effect of such convective heat transfer the difficulty may not be overcome.

This testifies to the possibility of preserving a

B

general concept developed for adiabatic flow, in this case the applicability of the assumption of linear entropy approximation, because the thermal effect appears to be insufficient for a qualitative change in the character of a process.

First of all, derive the equation, in which the relationship between stagnation and flow temperatures incorporates the above factors.

In [4] assuming that pv is a function of pT, i.e.

$$pv = \phi(p, T)$$

the expressions obtained may be presented as (in SI units)

$$w_1^2 - w_2^2 = 2\int_1^2\left[\left(\frac{\partial\varphi}{\partial p}\right)_T dp + \mu_T\frac{x}{x-1}R\,dT\right]$$

$$= 2\int_1^2\left(\frac{\partial\varphi}{\partial p}\right)_T dp + 2\bar{\mu}_T\cdot\frac{x}{x-1}R(T_2 - T_1) \qquad (1)$$

where the deviation ratio μ_T, μ_p and the quantity x introduced by A. M. Rosen [5] are accordingly expressed in the form

$$\left.\begin{aligned}\mu_T &= -\frac{p^2}{RT}\left(\frac{\partial v}{\partial p}\right)_T = z - p\left(\frac{\partial z}{\partial p}\right)_T\\ \mu_p &= \frac{p}{R}\left(\frac{\partial v}{\partial T}\right)_p = z + T\left(\frac{\partial z}{\partial T}\right)_p\end{aligned}\right\} \qquad (2)$$

$$x = \frac{c_p}{c_p - R\mu_p}. \qquad (3)$$

The right-hand side of expressions (2) is obtained by using the state equation for a real gas in the form

$$pv = zRT. \qquad (4)$$

The integral

$$\int_1^2\left(\frac{\partial\varphi}{\partial p}\right)_T dp$$

entering into expression (1) is written

$$\int_1^2\left[\frac{\partial(pv)}{\partial p}\right]_T dp.$$

To expand this integral, introduce the notation $\alpha = pv$. Taking into account that $pv = \varphi(p, T)$ it is possible to write

$$d\alpha = \left(\frac{\partial\alpha}{\partial p}\right)_T dp + \left(\frac{\partial\alpha}{\partial T}\right)_p dT.$$

Then

$$\int_1^2\left[\frac{\partial(pv)}{\partial p}\right]dp = \int_1^2 d\alpha - \int_1^2\left(\frac{\partial\alpha}{\partial T}\right)dT$$

$$= p_2v_2 - p_1v_1 - \int_1^2\left[\frac{\partial(pv)}{\partial T}\right]_p dT. \qquad (5)$$

Presenting expression (2) for μ_p in the form

$$\left[\frac{\partial(pv)}{\partial T}\right]_p = R\mu_p$$

it is possible to write

$$\int_1^2\left[\frac{\partial(pv)}{\partial T}\right]_p dT = R\bar{\mu}_p(T_2 - T_1) \qquad (6)$$

further, using the Lagrange formula upon transformations we have

$$(p_2v_2)_{T_2} - (p_2v_2)_{T_1} = (T_2 - T_1)R\mu_p.$$

Accordingly, substituting expression (6) into (5), upon a number of transformations, we arrive at

$$\int_1^2\left(\frac{\partial\varphi}{\partial p}\right)_T dp = \int_1^2\left[\frac{\partial(zRT)}{\partial p}\right]_T dp$$

$$= RT_1(z_2 - z_1)_{T_1}. \qquad (7)$$

It is necessary to mention that here z_2 and z_1 represent z at different pressures p_2 and p_1 but at the same temperature, namely, at the initial temperature T_1.

Substituting expression (7) into (1) we have

$$w_2^2 - w_1^2 = 2\left[R\bar{\mu}_T\frac{x}{x-1}(T_1 - T_2) - RT_1(z_2 - z_1)_{T_1}\right]. \qquad (8)$$

For jet stagnation we have $T_2 = T_0$, $z_2 = (z_0)_T$ and the jet velocity decreases from $w_1 = w$ (in this case $T_1 = T$, $z_1 = z$) to $w_2 = 0$. Expression (8) is then given as

$$\bar{\mu}_T \cdot \frac{x}{x-1} R(T_0 - T) + RT(z_0 - z)_T = \frac{w^2}{2} \quad (9)$$

$$w^2 = \lambda^2 a_{cr}^2. \quad (10)$$

Earlier we obtained expression [6]

$$a_{cr}^2 = \xi_{cr}^2 \cdot 2 \frac{k}{k+1} RT_0 \quad (11)$$

where

$$\xi_{cr}^2 = y_{cr}^2 \cdot v_{cr}^2$$

$$\xi_{cr}^2 = y_{cr}^2 \cdot \frac{k+1}{2} \left\{ \frac{1}{\bar{\mu}_T} \cdot \frac{x-1}{x} \times \left[\frac{k}{2} y_{cr}^2 + (z_{cr} - z_0)_{T_{cr}} \right] + 1 \right\}^{-1} \quad (12)$$

$$y_{cr}^2 = z_{cr}^2 \left\{ k \left[(\mu_T)_{cr} - \frac{x-1}{x} (\mu_p)_{cr} \right] \right\}^{-1}. \quad (13)$$

Solving expressions (9)–(11) simultaneously, upon transformations, we have

$$\frac{T_0}{T} = \left[1 + \frac{1}{\bar{\mu}_T} \cdot \frac{x-1}{x} (z - z_0)_T \right] \times \left(1 - \frac{k}{k+1} \cdot \xi_{cr}^2 \cdot \frac{1}{\bar{\mu}_T} \cdot \frac{x-1}{x} \cdot \lambda^2 \right)^{-1} \quad (14)$$

Taking into account the fact that in the presence of heat transfer the stagnation temperature is a variable quantity (T'_0) and introducing the notation

$$\left. \begin{aligned} b &= \frac{1}{\bar{\mu}_T} \cdot \frac{x-1}{x} (z - z_0)_T \\ c &= \frac{1}{\bar{\mu}_T} \cdot \xi_{cr}^2 \cdot \frac{k}{k+1} \cdot \frac{x-1}{x} \end{aligned} \right\} \quad (15)$$

expression (14) is written as

$$\frac{T'_0}{T} = \frac{1+b}{1 - c\lambda^2}. \quad (16)$$

On the basis of the first law of thermodynamics it is possible to write down

$$T\mathrm{d}S = c_v \mathrm{d}T + T\left(\frac{\partial p}{\partial T}\right)_v \mathrm{d}v. \quad (17)$$

With the help of coefficient [5]

$$\mu_v = \frac{T}{p} \left(\frac{\partial p}{\partial T}\right)_v \quad (18)$$

expression (17) is presented thus

$$T\mathrm{d}S = c_v \,\mathrm{d}T + \mu_v p \mathrm{d}v. \quad (19)$$

Using expression (4) and taking into account that $\mathrm{d}v/v = -\,\mathrm{d}\rho/\rho$ from expression (19) we have

$$\mathrm{d}S = c_v \frac{\mathrm{d}T}{T} - Rz\mu_v \frac{\mathrm{d}\rho}{\rho}. \quad (20)$$

Using the concept of the reduced entropy [1,2]

$$\sigma = \frac{S}{R} \quad (21)$$

and bearing in mind that

$$\mathrm{d}\sigma = \frac{\mathrm{d}S}{R} = \frac{\mathrm{d}L_{mp} \pm \mathrm{d}q}{RT} \quad (21')$$

as well as taking into account the relationship between c_p and c_v in the form [5]

$$c_p - c_v = R\mu_v^2 \mu_T = R \frac{\mu_p^2}{\mu_T} \quad (22)$$

where

$$\mu_v = \frac{\mu_p}{\mu_T}$$

it is possible to write

$$\mathrm{d}\sigma = \left(\frac{c_p}{R} - \frac{\mu_p^2}{\mu_T} \right) \frac{\mathrm{d}T}{T} - z \frac{\mu_p}{\mu_T} \frac{\mathrm{d}\rho}{\rho}. \quad (23)$$

Bearing in mind that mass flow rate ($\rho wF =$ const) is constant when

$$\frac{\mathrm{d}\rho}{\rho} = -\left(\frac{\mathrm{d}F}{F} + \frac{\mathrm{d}w}{w} \right) \quad (24)$$

it is possible to write

$$d\sigma = \left(\frac{c_p}{R} - \frac{\mu_p^2}{\mu_T}\right)\frac{d\tau}{\tau} + z\frac{\mu_p}{\mu_T}\left(\frac{df}{f} + \frac{dw}{w}\right). \tag{25}$$

Expression (11) for the square of a critical velocity of a real gas also contains a stagnation temperature T_0. Bearing in mind the presence of heat transfer, the stagnation temperature will be variable. Designating a variable stagnation temperature through T'_0 on the basis of expressions (10) and (11) it is possible to write down

$$\frac{dw}{w} = \frac{da_{cr}}{a_{cr}} + \frac{d\lambda}{\lambda} = \frac{1}{2}\frac{dT'_0}{T'_0} + \frac{d\xi_{cr}}{\xi_{cr}} + \frac{d\lambda}{\lambda}. \tag{26}$$

Introducing the notation $\theta = T'_0/T_0$ and giving expression (25) in dimensionless parameters, we have

$$d\sigma = d\ln\left[(f\lambda\xi_{cr})^{\frac{z\mu_p}{\mu_T}} \cdot \theta^{\frac{z\mu_p}{2\mu_T}} \cdot \tau^{c_p/R - \mu_p^2/\mu_T}\right]$$

$$= d\ln\psi = \frac{d\psi}{\psi} \tag{27}$$

where [2]

$$\psi = (f\lambda\xi_{cr})^{\frac{z\mu_p}{\mu_T}} \cdot \theta^{\frac{z\mu_p}{2\mu_T}} \cdot \tau^{\frac{c_p}{R} - \frac{\mu_p^2}{\mu_T}}. \tag{28}$$

It was earlier obtained [7] that

$$a^2 = y^2 kRT. \tag{29}$$

Obviously

$$a_{cr}^2 = y_{cr}^2 kRT_{cr}. \tag{30}$$

Solving expressions (11) and (30) simultaneously and taking into account heat transfer, i.e. taking a variable stagnation temperature T'_0 we obtain the relation

$$\frac{T_{cr}}{T'_0} = \frac{2}{k+1}\left(\frac{\xi_{cr}}{y_{cr}}\right)^2. \tag{31}$$

Substituting a value of T'_0 into expression (16) and then introducing the expression for τ into (28) also using (15) we have

$$\psi = (f\lambda\xi_{cr})^{\frac{z\mu_p}{\mu_T}} \cdot \theta^{\frac{z\mu_p}{2\mu_T}}$$

$$\times \left\{\frac{k+1}{2}\left(\frac{y_{cr}}{\xi_{cr}}\right)^2\left(1 - \frac{k}{k+1}\cdot\frac{x-1}{x}\right.\right.$$

$$\left.\left.\times \frac{\xi_{cr}}{\bar{\mu}_T}\cdot\lambda^2\right)\left[1 + \frac{1}{\bar{\mu}_T}\right.\right.$$

$$\left.\left.\times \frac{x-1}{x}(z - z_0)_T\right]^{-1}\right\}^{\frac{c_p}{R} - \frac{\mu_p^2}{\mu_T}} \tag{32}$$

Owing to the linearity hypothesis accepted it is assumed [2]

$$\frac{d\sigma}{dl} = \text{const.}$$

Dividing both hand-sides of expression (27) into into $d\bar{l} = dl/D$ and introducing the notation

$$\eta = \frac{d\sigma}{d\bar{l}} = \frac{d\ln\psi}{d\bar{l}} \tag{33}$$

we obtain [2]

$$\ln\psi = \eta l$$

or

$$\varphi = l^{\eta\bar{l}}. \tag{34}$$

For isentropic gas flow $\phi = \text{const.} = 1$. In our case at $T_0 = T'_0$ or $\theta = 1$ (no heat transfer in isentropic flow).

Then from equation (28) we have*

$$f_s = \left[\lambda\cdot\xi_{cr}\cdot\tau^{\frac{\mu_T}{2\mu_p}\left(\frac{c_p}{R} - \frac{\mu_p^2}{\mu_T}\right)}\right]^{-1}. \tag{35}$$

With regard for expression (35) expression (28) may be given in the form

$$\psi = \left(\frac{f}{f_s}\right)^{\frac{z\mu_0}{\mu_T}} \cdot \theta^{\frac{1}{2}\cdot\frac{z\mu_p}{2\mu_T}}. \tag{36}$$

Solving expressions (36) and (34) simultaneously, we have

$$f_s = f\cdot\theta^{\frac{1}{2}}\cdot e^{\left(-\eta\bar{l}\cdot\frac{\mu_T}{z\mu_p}\right)}. \tag{37}$$

Expression (37) may serve as the basis of gas dynamic calculations of flow of a real gas in a channel in the presence of heat transfer.

In gas-dynamic calculations of nozzles or cooled gas-turbine blades when a gas is considered to be ideal due to the presence of high

* The subscript s denotes isentropic flow conditions.

temperatures we obtain in the presence of heat transfer

$$(f_s)_{\text{ideal}} = f_{\text{ideal}} \cdot \theta^{\frac{1}{2}} \cdot e^{-\eta \bar{l}} \tag{38}$$

For an ideal gas with no heat transfer ($\theta = 1$) we have the relation as that obtained in [2]

$$(f_s)_{\text{ideal}} = f_{\text{ideal}} \cdot e^{-\eta \bar{l}}. \tag{39}$$

In conclusion, note that for gas-dynamic calculations of flow of a real gas involving heat transfer it is not enough to have the law of change of cross-sectional area as in case of an ideal gas flow [2].

In the case under consideration in equation (37) the presence of the expression $\mu_T / \cdot z\mu_p$ being a function of p and T, on the one hand, and a quantity θ equal to T'_0/T, on the other hand, causes the necessity to have a law of a pressure and temperature distribution along a channel length when making calculations.

Equations (37)–(39) allow the following important problems [2] to be solved:

1. By a prescribed value of η and a law of change in f along the channel length it is possible to determine the change in flow parameters along the channel under consideration;
2. when channel sizes, a value of η and coefficient λ are prescribed it is possible to find a section corresponding to a prescribed value of λ.

It stands to reason that calculations incorporating properties of a real gas and heat transfer will be complicated and, consequently, for the problem stated to be solved it is necessary to use the method of successive approximations.

ACKNOWLEDGEMENT

The author wishes to thank Prof. A. A. Gukhman for interesting discussions of the problems considered in the present paper.

REFERENCES

1. A. A. GUKHMAN, A. F. GANDELSMAN and N. V. ILYUKHIN, Study of a change in the resistance coefficient for a gas flowing with supersonic velocity, *Teploenergetika* No. 1, 17–23 (1955).
2. A. A. GUKHMAN, A. F. GANDELSMAN and I. N. NAURITS, On hydrodynamic resistance in the supersonic flow region, *Energomashinistroyeniye* No. 7, 10–14 (1957).
3. A. A. GUKHMAN, On the theory of limiting states of a moving gas, *Zh. Tekh. Fiz.* **IX**, 411–423 (1939).
4. E. A. ORUDZHALIEV, On the theory of shock waves in the dynamics of a real gas, *Int. J. Heat Mass Transfer* **6**, 935–940 (1963).
5. A. M. ROSEN, The method of deviation coefficients in thermodynamics of high pressures, *Zh. Fiz. Khim.* **19**, 469–484 (1945).
6. E. A. ORUDZHALIEV, General equations for real gas flow, *Izv. Vuzov Neft' i gaz* No. 5. 91–97 (1959).
7. E. A. ORUDZHALIEV, Velocity of sound of a real gas, *Izv. Vuzov Neft' i gaz* No. 8, 89–96 (1958).

QUELQUES RELATIONS POUR LA REGION SUPERSONIQUE D'ECOULEMENT D'UN GAZ REEL EN PRESENCE D'UN TRANSFERT THERMIQUE

Résumé—Les relations pratiques pour l'écoulement d'un gaz réel en présence d'un transfert thermique sont dérivées de la méthode entropique de A. A. Gukhman.

BEZIEHUNGEN FÜR DEN ULTRASCHALLBEREICH DER STRÖMUNG EINES REALEN GASES MIT WÄRMEÜBERGANG

Zusammenfassung—Die Beziehungen zur Berechnung der Strömung eines realen Gases mit Wärmeübergang sind auf Grund der Entropie-Methode von A. A. Gukhman abgeleitet.

НЕКОТОРЫЕ СООТНОШЕНИЯ В ТРАНСЗВУКОВОЙ ОБЛАСТИ ТЕЧЕНИЯ РЕАЛЬНОГО ГАЗА ПРИ НАЛИЧИИ ТЕПЛООБМЕНА

Аннотация—В работе на основе энтропийного метода А.А. Гухмана выведены расчетные соотношения в течении реального газа при наличии теплоobмена.